Self-interference of a single Bose-Einstein condensate due to boundary effects

R. W. Robinett*

Department of Physics

The Pennsylvania State University

University Park, PA 16802 USA

(Dated: February 1, 2008)

Abstract

A simple model wavefunction, consisting of a linear combination of two free-particle Gaussians, describes many of the observed features seen in the interactions of two isolated Bose-Einstein condensates as they expand, overlap, and interfere. We show that a simple extension of this idea can be used to predict the qualitative time-development of a single expanding BEC condensate produced near an infinite wall boundary, giving similar interference phenomena. We also briefly discuss other possible time-dependent behaviors of single BEC condensates in restricted geometries, such as wave packet revivals.

PACS numbers: 03.75.Kk, 03.75.Hh, 03.75.Nt, 03.65.Ge

*Electronic address: rick@phys.psu.edu

I. INTRODUCTION

It can be argued that much of the early success of quantum theory can be traced to the fact that many exactly soluble quantum models are surprisingly coincident with naturally occurring physical systems, such as the hydrogen atom and the rotational/vibrational states of molecules. Many other exemplary quantum mechanical models, which have historically been considered as only textbook idealizations, have also recently found experimental realizations. Advances in areas such as materials science or laser trapping and cooling of atoms have allowed the production of approximations to a number of systems which had typically been relegated to lists of pedagogical examples. While some such examples have found use as devices, many others have been applied to the study of fundamental quantum behavior.

For example, one-dimensional quantum wells (infinite and finite) are a staple of textbooks and have found use in modeling quantum dots and other structures [1], including the use of asymmetric wells [2]. Two-dimensional quantum mechanical 'standing waves' have been observed in a wide variety of geometries [3], while evidence for bound quantum states of the neutron in the Earth's gravitational field [4] (a problem which is often described as the 'quantum bouncer' in the pedagogical literature) has recently been presented. The *"generation of nonclassical motional states"*, such as coherent and squeezed states [5] of harmonically trapped ions, has also been demonstrated and studied in detail as an example of time-dependent designer wave functions in the most familiar of all model potentials.

The experimental realization of Bose-Einstein condensates (BECs) [6] has allowed for an even wider variety of fundamental tests, including the "*Observation of interference between two Bose condensates*" [7]. In the original experiment [7], two samples of sodium atoms were evaporatively cooled "*well below the transition temperature to obtain condensates*" such that they were initially well-separated in a double-well potential. The two condensates were then allowed to freely expand and interference effects were observed in the resulting overlap region, while no similar effects were seen for a single expanding condensate. Similar effects have been seen in other experimental realizations [8] and more recently have been observed with up to 30 uncorrelated condensates [9] produced in an optical trap.

As we will briefly review in Sec. II, a simple model wavefunction consisting of a linear combination of two [10] (or more [9]) Gaussian terms, $\psi_{(G)}(x,t;x_0)$, (one for each condensate) captures many of the salient features observed experimentally. A (single particle)

wavefunction of the form

$$\psi(x,t) = N\left[\psi_{(G)}(x,t;x_A) + e^{i\phi}\psi_{(G)}(x,t;x_B)\right] \tag{1}$$

can be used, with $x_0 = x_A, x_B$ describing the locations of the two isolated condensates and with fixed relative phase [11] (and a normalization constant N) and we will discuss the predictions of this simple model in the next section.

Such linear combination solutions (especially of Gaussians) have been frequently used in the pedagogical literature to describe the time-development of wave packet solutions of the 1D Schrödinger equation (SE) describing an otherwise free-particle impinging on an infinite wall or barrier (or "free particle on the half-line".) "Mirror" or "image" solutions of the form

$$\tilde{\psi}(x,t) = \begin{cases} \tilde{N}\left[\psi(x,t) - \psi(-x,t)\right] & \text{for } x \leq 0 \\ 0 & \text{for } 0 \leq x \end{cases} \tag{2}$$

(for a particle restricted to $x \leq 0$) satisfy the 1D free-particle SE (if $\psi(x,t)$ does) and also automatically satisfy the appropriate boundary condition at the infinite wall (assumed to be at $x = 0$) for all times. Such solutions have been used in a variety of pedagogical applications [12] – [14], but also in a research context to discuss the deflection of ultracold quantum particles (wavepackets) from impenetrable boundaries or mirrors [15]. Such analyses naturally explain the spatially oscillatory behavior of the position-space probability density as the wave packet 'hits' the wall (as observed in numerical calculations) as the interference between two overlapping terms, much like the observed BEC effect.

In this Letter, we note that a single BEC condensate, produced near an infinite wall boundary and allowed to expand freely, will likely exhibit interference effects describable by such 'mirror' or 'image' solutions as in Eqn. (2), and we discuss this in Sec. III. We also extend such ideas to a localized BEC produced between two infinite boundaries (as in an infinite well potential) to very briefly discuss other possible effects, such as wave packet revivals.

II. SIMPLE MODEL OF INTERFERING BOSE-EINSTEIN CONDENSATES

A description of the initial (single particle) one-dimensional wavefunction for two separated BEC condensates using isolated Gaussian forms has been made by the authors of

Ref. [10], who then analyze the resulting time-development of the BECs using a Wigner distribution approach. (A generalization to multiple BECs is given in Ref. [9].) As a review of such an approach and its successes in qualitatively modeling more sophisticated analyses (and the experimental data), consider two BEC condensates, separated by a distance d and initially centered at $x_0 = \pm d/2$, described by the time-dependent free-particle wavefunction

$$\psi_{2BEC}(x,t) = \frac{N}{\sqrt{\sqrt{\pi}\beta(1+it/t_0)}} \left[e^{-(x-d/2)^2/2\beta^2(1+it/t_0)} + e^{i\phi} e^{-(x+d/2)^2/2\beta^2(1+it/t_0)} \right] \quad (3)$$

where the normalization factor is given by

$$N = \frac{1}{\sqrt{2}} \left(1 + \cos(\phi) e^{-d^2/4\beta^2} \right)^{-1/2} . \quad (4)$$

The time-dependent spatial width for a *single* such Gaussian term is given by

$$\Delta x_t = \frac{\beta_t}{\sqrt{2}} \equiv \frac{\beta}{\sqrt{2}} \sqrt{1 + (t/t_0)^2} \qquad \text{where} \qquad t_0 \equiv \frac{m\beta^2}{\hbar} \quad (5)$$

and the corresponding spread in momentum-space (again, for a single Gaussian) is $\Delta p_t = \Delta p_0 = \hbar/(\beta\sqrt{2})$. The time-dependent probability density for the two condensate state is then given by

$$P_{2BEC}(x,t) = \frac{N^2}{\sqrt{\pi}\beta_t} \left[e^{-(x-d/2)^2/\beta_t^2} + e^{-(x+d/2)^2/\beta_t^2} + 2e^{-(d^2+4x^2)/4\beta_t^2} \cos\left(\phi + \frac{tdx}{t_0 \beta_t^2} \right) \right] \quad (6)$$

where the cross-term describes the interference effect.

For each Gaussian contribution, there are momentum components of order $p \sim \Delta p_0 = \hbar/\beta\sqrt{2}$ so that the time (T_O) it takes such components to drift from one condensate and overlap with the other is of order $T_O \sim d/(p/m) \sim \sqrt{2} dm\beta/\hbar$. For condensates which are initially highly localized and well-separated, the time dependent position-spread is then dominated by the $(t/t_0)^2$ term since $(T_O/t_0)^2 \sim 2(d/\beta)^2 >> 1$. In that limit, the oscillatory term is then approximately given by

$$\cos\left(\phi + \frac{xdt}{\beta_t^2 t_0} \right) \qquad \longrightarrow \qquad \cos\left(\phi + \frac{xdt_0}{\beta^2 t} \right) \quad (7)$$

so that the local wavelength variations seen in the interference pattern are time-dependent and scale like

$$\frac{2\pi}{\lambda} x = kx = \frac{xdt_0}{\beta^2 t} \qquad \text{or} \qquad \lambda = \frac{ht}{md} , \quad (8)$$

just as in Eqn. (1) of Ref. [7]. The time-dependent real and imaginary parts of the individual components of this simple wavefunction are nicely consistent with more detailed

calculations [8] where the BEC is "*characterized by a phase that varies quadratically ...across the condensate*"; for example, compare the pedagogical illustration of an expanding $p = 0$ Gaussian wavepacket in Fig. 2 of Ref. [16] with Fig. 1 of Ref. [8]. Finally, the corresponding momentum-space probability density is given by

$$|\phi_{2BEC}(p,t)|^2 = \frac{4N^2\alpha}{\sqrt{\pi}} \cos^2\left(\frac{pd}{2\hbar}\right) e^{-\alpha^2 p^2} \tag{9}$$

where $\alpha \equiv \beta/\hbar$ so that there is indeed structure in momentum space at integral multiples of $p = h/d$, as discussed in more detail in Ref. [17]. Thus, in many important ways, the simple wavefunction in Eqn. (3) encodes much of the physics observed in the interference of two expanding BEC condensates.

III. SINGLE BEC NEAR A INFINITE WALL BOUNDARY AND RELATED EFFECTS

Motivated then by earlier pedagogical papers on 'mirror' or 'image' solutions [12] – [14], we can imagine a single BEC condensate produced close to an infinite barrier. If the barrier is located at $x = 0$ and the single condensate is produced at $x = -d/2$, the resulting 'mirror' solution in Eqn. (2), for $x \leq 0$, can be described by the form in Eqn. (3) with $\phi = \pi$ (so that $\cos(\phi) = -1$) and with normalizations simply related by $\tilde{N} = \sqrt{2}N$. The resulting time-dependent solution will then exhibit the same type of interference patterns observed for two isolated condensates. One possibility for such an infinite barrier might be an atomic mirror [18] of the type successfully used in a number of atomic physics applications [19].

The addition of a second infinite wall barrier (say at $x = -d$) to such a case might then be modeled by the standard infinite well problem of textbooks. The time development of wave packet propagation in this system can then be described in terms of an infinite number of image solutions [20] and discussions related to this approach go back to at least Einstein and Born [21]. For a single BEC condensate in this restricted geometry, modeled as a $p = 0$ Gaussian, in addition to the spreading/coherence time (t_0) and the time to overlap the other real or image condensate (T_O), the only other relevant time scale is the quantum revival time [22] T_{rev}. For the quantized energy eigenvalues in an infinite well of width d (as imagined here) the revival time for an arbitrary localized wave packet is $T_{rev} = 4md^2/\hbar\pi$ [23]. The ratio of revival time to overlap time is then $T_{rev}/T_O \sim 2d/\pi\Delta x_0 >> 1$. In the original two

BEC experiment [7], the two condensates are allowed to fall freely as they expand, so a more detailed analysis of a specific experiment realization would be required to determine if the revival time is too long to be observed. One should note, however, that for the special case of a $p = 0$ Gaussian waveform produced precisely in the center of such an infinite well, because only even energy eigenstates are excited, the effective revival time is actually $T_{rev}/8$ [22] for this very special geometry, which gives almost an order-of-magnitude shorter fall time to achieve a revival.

Other arrangements of two infinite plane barriers can also be imagined to give rise to interesting BEC interference effects, which can be modeled using 'mirror' or 'image' methods. For example, two such infinite walls can be placed at right angles, to form a 'corner (90°) reflector', defined by the potential

$$V(x,y) = \begin{cases} 0 & \text{for } 0 < x \ \underline{\textbf{and}} \ 0 < y \\ +\infty & \text{otherwise} \end{cases} . \tag{10}$$

A solution of the form

$$\psi_{corner}(x,y;t) = N\left[\psi(x,y;t) - \psi(-x,y;t) - \psi(x,-y;t) + \psi(-x,-y;t)\right] \tag{11}$$

making use of three auxiliary 'image' components, solves the Schrödinger equation in the allowed region for any free-particle solution $\psi(x,y;t)$, as well as satisfying the boundary conditions at the two walls. The normalization factor can be obtained explicitly in the case of an Gaussian solution. as above. The same construction can also be employed for other angles between the two walls, Θ, using familiar examples from optics, such as for the cases of $\Theta = 45°$ and $60°$.

---

[1] Masumoto, Y. and Takagahara, T. (Eds.), *Semiconductor Quantum Dots: Physics Spectroscopy, and Applications* (Springer, Berlin, 2002); Michler, P. (Ed.), *Single Quantum Dots: Fundamentals, Applications, and New Concepts*, (Springer, Berlin, 2003); Borovitskaya, E. and Shur, M. S. (Eds.), *Quantum Dots*, (World Scientific, New Jersey, 2002).

[2] Bonvalet, A., Nagle, J., Berger, V., Migus, A., Martin, J. L., Joffre, M., Phys. Rev. Lett. **76**, 4392 (1996); Harrison, P., *Quantum Wells, Wires, and Dots: Theoretical and Computational Physics*, (Wiley, Chichester, 1999).

[3] Crommie, M. F., Lutz, C. P., and Eigler, D. M., Nature **363**, 524, (1993); Science **262**, 218 (1993); Heller, E. J., Crommie, M. F., Lutz, C. P., and Eigler, D. M., Nature **369**, 464 (1994); Crommie, M. F., Lutz, C. P., Eigler,D. M., and Heller, E. J., Surf. Rev. Lett. **2**, 127 (1995); Physica D **83**, 98 (1995).

[4] Nesvizhevsky, V. V. *al.*, Nature **415**, 297 (2002); Nesvizhevsky, V. V. *al.*, Phys. Rev. D **67**, 102002 (2003); Phys. Rev. D **68**, 108702 (2003).

[5] Meekhof, D. M., Monroe, C., King, B. E., Itano,W. M., and Wineland, D.J., Phys. Rev. Lett. **76**, (1996) 1796; *ibid.* **77**, 2346 (1996).

[6] For a review, see Pitaevskii, L. and Stringari, S., *Bose-Einstein Condensation* (Clarendon Press, Oxford, 2003).

[7] Andrews, M. R., Townsend, C. G., Miesner, H. -J., Durfee, D. S., Kurn, D. M., and Ketterle, W., Science **275**, 637 (1997); Durfee, D. S. and Ketterle, W., Optics Express **2**, 299 (1998).

[8] Hagley, E. W. *et al.* Phys. Rev. Lett. **83**, 3112 (1999).

[9] Hadzibabic, Z., Stock, S., Battelier, B., Bretin, V., and Dalibard, J., Phys. Rev. Lett. **93**, 180403 (2004).

[10] Wallis, H., Röhrl, A., Naraschewski, M., and Schenzle, A., Phys. Rev. **A55**, 2109 (1997).

[11] Anderson, P. W., in *The Lesson of Quantum Theory*, Proceedings of the Niels Bohr Centenary Symposium, Copenhagen, Denmark, 1985, J. D. Boer, E. Dal and O. Ulfbeck (Eds.) (Elsevier, Amsterdam, 1986), p. 23.

[12] Andrews, M., Am. J. Phys. **66**, 252 (1998).

[13] Doncheski, M. A. and Robinett, R. W., Eur. J. Phys. **20**, 29 (1999); Belloni, M., Doncheski, M. A., and Robinett, R. W., Phys. Scripta **71**, 136 (2005).

[14] Thaller, B., *Visual Quantum Mechanics: Selected Topics with Computer-Generated Animations of Quantum-Mechanical Phenomena*, (Springer/TELOS, New York, 2000).

[15] Dodonov, V. V. and Andreata, M. A., Phys. Lett. **A275**, 173 (2000); Laser Phys. **12**, 57 (2002); Andreata, M. A. and Dodonov, V. V., J. Phys. Math. A: Math. Gen. **35**, 8373 (2002).

[16] Robinett, R. W. and Bassett, L. C., Found. Phys. Lett. **17**, 607 (2004).

[17] Pitaevskii, L. and S. Stringari, S., Phys. Rev. Lett. **83**, 4237 (1999).

[18] Dowling, J. P. and Gea-Banacloche, J., Adv. At. Mol. Opt. Phys. **37**, 1 (1996).

[19] Aminoff, C. G., Steane, A. M., Bouyer, P., Desbiolles, P., Dalibard, J., and Cohen-Tannoudji, C., Phys. Rev. Lett. **71**, 3083 (1993); Roach, T. M., Abele, H., Boshier, M. G., Grossman, H.

L., Zetie, K. P., and Hinds, E. A., Phys. Rev. Lett. **75**, 629 (1995); Landragin, A., Courtois, J. Y., Labeyrie, G., Vansteenkiste, N., Westbrook, C. I., and Aspect, A., Phys. Rev. Lett. **77**, 1464 (1996); Szriftgiser, P., Guëry-Odelin, D., Arndt, M., and Dalibard, J., Phys. Rev. Lett. **77**, 4 (1996).

[20] Kleber, M., *Exact solutions for time-dependent phenomena in quantum mechanics*, Phys. Rep. **236**, 331 (1994); Stifter, P., Leichtle, C., Schleich, W. P., and Marklov, J., *Das Teilchen im Kasten: Strukturen in der Wahrscheinlichkeitsdichte* (translated as *The particle in a box: Structures in the probability density*), Z. Naturforsch **52a**, 377 (1997); Aronstein, D. L. and Stroud, C. R., Jr., Phys. Rev. **A55**, 4526 (1997).

[21] Born, M., *Continuity, determinism, and reality*, Kgl. Danske Videns. Sels. Mat.-fys. Medd., **30** (2), 1 (1955). Born was addressing concerns made by Einstein in *Elementare Überlegungen zur Interpretation der Grundlagen der Quanten-Mechanik*, in *Scientific papers presented to Max Born* (Oliver and Boyd, Edinburgh, 1953) pp. 33-40; Born, M. and Ludwig, W., *Zur Quantenmechanik der kräftefreien Teilchens*, Z. Phys. **150**, 106 (1958).

[22] For a recent review of quantum wave packet revivals, see Robinett, R. W., Phys. Rep. **392**, 1 (2004).

[23] Segre, C. U. and Sullivan, J. D., Am. J. Phys. **44**, 729 (1976). This pedagogical paper contains one of the first references to what have become known as quantum revivals, examined in the context of the infinite well.